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Journal of Approximation Theory 125 (2003) 151–168

JOURNAL OF
Approximation
Theory

http://www.elsevier.com/locate/jat

On the holes of a class of bidimensional nonseparable wavelets☆

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Received 24 May 2002; accepted in revised form 3 November 2003

Communicated by Zuowei Shen

Abstract

Let I be the 2×2 identity matrix, and M a 2×2 dilation matrix with $M^2 = 2I$. Since one can explicitly construct M-basic wavelets from an MRA related to M, and many applications employ wavelet bases in R^2, M-wavelets and wavelet frames have been extensively discussed. This paper focuses on dilation matrices M satisfying $M^2 = 2I$. For any matrix M integrally similar to $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, an optimal estimate on the boundary of the holes of M-wavelets is obtained. This result tells us the holes cannot be too large. Contrast to this result, when the modulus of the Fourier transform of an M-wavelet is, up to a constant, a characteristic function on some set, a property of this set is obtained, which shows the holes of this kind of wavelets cannot be too small.

MSC: 42C40

Keywords: Wavelet; Wavelet frame; Integral similarity

1. Introduction

Throughout this paper, we denote by Z the set of integers, Z_+ the set of nonnegative integers, and I the 2×2 identity matrix. For $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in R^2$,

☆Supported by the Educational Committee Foundation of Beijing(01KJ-019).
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0021-9045/$ - see front matter
doi:10.1016/j.jat.2003.11.002

$\langle x, y \rangle$ is defined by

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2.$$

For any Lebesgue measurable set $S \subset R^2$ and 2×2 matrix A, $|S|$ denotes the Lebesgue measure of S, χ_S denotes the characteristic function on S, A^* denotes the transpose of A, and AS denotes the set

$$AS = \{Ax\colon\ x \in S\}.$$

For any function f defined on R^2, supp(f) is defined by

$$\text{supp}(f) = \{x \in R^2\colon\ f(x) \neq 0\}.$$

For $f \in L^1(R^2)$, we define the *Fourier transform* of f by

$$\hat{f}(\cdot) = \frac{1}{2\pi} \int_{R^2} dx\, f(x) e^{-i\langle x, \cdot \rangle}.$$

Let S be a Lebesgue measurable set in R^2. A collection $\{S_\gamma\colon\ \gamma \in \Gamma\}$ of Lebesgue measurable subsets of S is called a *partition* of S if $\bigcup_{\gamma \in \Gamma} S_\gamma = S$ up to a set of measure 0, and $|S_\gamma \cap S_{\gamma'}| = 0$ for $\gamma \neq \gamma'$, γ, $\gamma' \in \Gamma$.

A 2×2 matrix M is called a *dilation matrix* if it is an integer matrix with its all eigenvalues λ's being larger than 1 in modulus.

Let M be a given 2×2 dilation matrix with $M^2 = 2I$. For any function f defined on R^2, $j \in Z$, and $k \in Z^2$, define

$$f_{j,k}(\cdot) = 2^{-\frac{j}{2}} f(M^{-j} \cdot - k).$$

A function ψ is called an M-*wavelet* if $\{\psi_{j,k}\colon\ j \in Z, k \in Z^2\}$ is an orthonormal basis for $L^2(R^2)$; and the *hole* of the M-wavelet is referred to be the maximum connected set containing a neighborhood of the origin on which $\hat{\psi}$ vanishes a.e. A function collection $\psi^1, \psi^2, \ldots, \psi^r$ is called to *generate an M-tight frame with frame bound* 1 for $L^2(R^2)$ if

$$\sum_{l=1}^{r} \sum_{j \in Z, k \in Z^2} |\langle f, \psi^l_{j,k} \rangle|^2 = ||f||_2^2$$

for $f \in L^2(R^n)$. A ladder of closed subspaces $\{V_j\}_{j \in Z}$ of $L^2(R^2)$ is called a *multiresolution analysis* (MRA) related to M if the following conditions hold:

(1) $V_j \subset V_{j-1}$ for $j \in Z$;
(2) $\bigcap_{j \in Z} V_j = \{0\}$, $\overline{\bigcup_{j \in Z} V_j} = L^2(R^n)$;
(3) $f(\cdot) \in V_j$ if and only if $f(M^j \cdot) \in V_0$ for $j \in Z$;
(4) there exists a function $\phi(\cdot)$ in V_0 such that the set $\{\phi(\cdot - k)\}_{k \in Z^2}$ is an orthonormal basis for V_0.

It is well known that one can obtain an M-wavelet from an MRA related to M, but not any M-wavelet can be derived from an MRA related to M [6,9].

In recent years, multi-dimensional wavelets and wavelet frames have been extensively discussed [2–4,7,10–12,15–17,19,20]. Especially, the study of bidimensional wavelets has attracted many mathematician's interest. Many applications, such as image compression, employ wavelet bases in R^2. Although separable bases have a lot of advantages, they have a number of drawbacks. They are so special that they have very little design freedom, and separability imposes an unnecessary product structure on the plane, which is artificial for natural images. This preferred directions effect can create unpleasant artifacts that become obvious at high image compression ratios. Nonseparable wavelet bases offer the hope of a more isotropic analysis [1,8,13,14,21]. M-wavelets are nonseparable. Hence, one may hope for a more isotropic analysis than with the separable construction. This is also why we deliberately restrict ourselves to the matrix M.

One-dimensional analyzing wavelets ψ's in continuous wavelet transforms are required to satisfy the admissibility condition, which leads to

$$\hat{\psi}(0) = 0$$

if $\hat{\psi}$ is also continuous at the origin. This is the case since $\psi \in L^1(R)$ in almost all examples of practical interest. It is also well known that if ψ is a 2-wavelet and $\hat{\psi}$ is continuous at the origin, then

$$\sum_{j \in Z} |\hat{\psi}(2^{-j}\cdot)|^2 = 1$$

a.e., which also implies that

$$\hat{\psi}(0) = 0.$$

Hence, it is reasonable for a 2-wavelet ψ to require that $\hat{\psi}(0) = 0$, and consequently, it is interesting to introduce the concept of the hole of a wavelet. In [5], the behavior at the origin of a class of band-limited 2-wavelets is deliberately addressed, and many sharp results are obtained. Obviously the set $[-\pi, \pi]$ is the hole of the shannon 2-wavelet. The size of the hole of a 2-wavelet is discussed in [18]. By Theorems 5 and 6 in [18], we have that

Proposition 1. *Let ψ be a 2-wavelet for $L^2(R)$. Then $|\mathrm{supp}(\hat{\psi}) \cap [-c, c]| > 0$ for $c > \pi$.*

Proposition 2. *Let ψ be a 2-wavelet for $L^2(R)$ with $|\hat{\psi}| = \frac{1}{\sqrt{2\pi}} \chi_K$ for some $K \subset [-2\pi, 2\pi]$. Then,*

$$|K \cap [-\pi, \pi]| = 0.$$

Analogously, suppose that ψ is an M-wavelet whose Fourier transform is continuous at the origin. Then,

$$\sum_{j \in Z} |\hat{\psi}((M^*)^{-j}\cdot)|^2 = 1$$

a.e., which implies that

$$\hat{\psi}(0) = 0.$$

Therefore, it is a natural problem to discuss the holes of M-wavelets.

Definition 1.1 (Bownik and Speegle [4], Lagarias and Wang [15]). Let A and B be two 2×2 matrices. A is called integrally similar to B if there exists an integer matrix C with $|\det C| = 1$ such that $CAC^{-1} = B$. Here we call C an integrally similar matrix from A to B.

Remark 1.1. Integrally similar matrix is not necessarily unique. For example, both $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$ are integrally similar matrix from $\begin{pmatrix} 2 & 1 \\ -2 & -2 \end{pmatrix}$ to $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

In this paper, we are concentrated on the study of the holes of M-wavelets, where M is a 2×2 matrix integrally similar to $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Our main results can be stated as follows.

Theorem 1.1. *Let M be a 2×2 dilation matrix, and M be integrally similar to the matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, i.e., $PMP^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ for some integer matrix P with $|\det P| = 1$. Suppose ψ is an M-wavelet, then,*

$$|\mathrm{supp}(\hat{\psi}) \cap P^*(\{(\xi_1, \xi_2)^T \colon |\xi_1| \leqslant c, \ |\xi_1 + \xi_2| \leqslant c\})| > 0,$$

$$|\mathrm{supp}(\hat{\psi}) \cap P^*(\{(\xi_1, \xi_2)^T \colon |\xi_2| \leqslant c, \ |\xi_1 - \xi_2| \leqslant c\})| > 0$$

for $c > \pi$.

Remark 1.2. Theorem 1.1 shows the holes of M-wavelets cannot be too large. In addition, the following Theorem 1.2 shows Theorem 1.1 is optimal in the following sense: it does not hold for $c = \pi$.

Theorem 1.2. *Let M be a 2×2 dilation matrix, and M be integrally similar to the matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, i.e., $PMP^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ for some integer matrix P with $|\det P| = 1$. Suppose ψ is an M-wavelet satisfying that $|\hat{\psi}| = \frac{1}{2\pi}\chi_K$ for some K satisfying*

$$K \subset P^*(\{(\xi_1, \xi_2)^T \colon |\xi_1| \leqslant \pi, \ |\xi_1 + \xi_2| \leqslant 2\pi\})$$

$$(\textit{or } K \subset P^*(\{(\xi_1, \xi_2)^T \colon |\xi_2| \leqslant \pi, \ |\xi_1 - \xi_2| \leqslant 2\pi\})),$$

then

$$|K \cap P^* S_\pi^{(1)}| = 0 \quad (\textit{or } |K \cap P^* S_\pi^{(2)}| = 0),$$

where

$$S_\pi^{(1)} = \{(\xi_1, \xi_2)^T \colon |\xi_1| \leqslant \pi, |\xi_1 + \xi_2| \leqslant \pi\},$$

$$S_\pi^{(2)} = \{(\xi_1, \xi_2)^T \colon |\xi_2| \leqslant \pi, |\xi_1 - \xi_2| \leqslant \pi\}.$$

Remark 1.3. Theorem 1.2 shows the holes of M-wavelet in this the theorem cannot be too small. An example of ψ satisfying the hypothesis of Theorem 1.2 is given in Example 4.1 in Section 4.

Remark 1.4. Note that M-wavelets are not separable, which is not the tensor product of one-dimensional wavelets. We have not known the results about the M-wavelet set as in one-dimensional case. So Theorem 1.2 is not the simple consequence of wavelet sets.

Remark 1.5. For a matrix integrally similar to $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, an integrally similar matrix P from M to $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ can be obtained according to the algorithm below Lemma 2.4 in Section 2.

Remark 1.6. By Remark 2.1 below Lemma 2.1 in Section 2 and the analogous argument to that at the beginning of Section 4, Theorem 1.1 holds under the following conditions: M satisfies $PMP^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ for some invertible matrix P; $\{2^{-\frac{j}{2}}\psi(M^{-\frac{j}{2}} \cdot - Pk) \colon j \in Z, k \in Z^2\}$ is an orthonormal basis for $L^2(R^2)$. If, in addition, $|\hat{\psi}| = \frac{1}{2\pi\sqrt{|\det P|}}\chi_K$, then Theorem 1.2 holds. Note that $PZ^2 = Z^2$ if P is a 2×2 integer matrix with $|\det P| = 1$. The two conclusions generalize Theorems 1.1 and 1.2, respectively.

In Section 2, some auxiliary lemmas and an algorithm of integrally similar matrix are given. In Section 3, some properties of $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$-wavelets are given. In Section 4, the proofs of the theorems are given, and an example of Theorem 1.2 is also given.

2. Some auxiliary lemmas and an algorithm of integrally similar matrix

Lemma 2.1. *Assume that $\psi \in L^2(R^2)$, that M_1 is a 2×2 dilation matrix with $M_1^2 = 2I$, and that M_2 is integrally similar to M_1, i.e., $M_2 = PM_1P^{-1}$ for $|\det P| = 1$. Then $\psi(\cdot)$ is an M_1-wavelet for $L^2(R^2)$ if and only if $\tilde{\psi}(\cdot) = \psi(P^{-1}\cdot)$ is an M_2-wavelet for $L^2(R^2)$.*

Proof. Note that $PM_1P^{-1} = M_2$, it is easy to check that

$$\begin{aligned}\int_{R^2} & dx\, 2^{-\frac{j}{2}}\tilde{\psi}(M_2^{-j}x-k)\overline{2^{-\frac{j'}{2}}\tilde{\psi}(M_2^{-j'}x-k')} \\ &= \int_{R^2} dx\, 2^{-\frac{j}{2}}\psi(P^{-1}M_2^{-j}x-P^{-1}k)\overline{2^{-\frac{j'}{2}}\psi(P^{-1}M_2^{-j'}x-P^{-1}k')} \\ &= \int_{R^2} dx\, 2^{-\frac{j}{2}}\psi(M_1^{-j}x-P^{-1}k)\overline{2^{-\frac{j'}{2}}\psi(M_1^{-j'}x-P^{-1}k')} \qquad (2.1)\end{aligned}$$

for $j, j' \in Z$, $k, k' \in Z^2$.

Suppose ψ is an M_1-wavelet for $L^2(R^2)$. Then, by (2.1), $\tilde{\psi}$ generates an M_2-orthonormal system. For any $f \in L^2(R^2)$, $f(P\cdot) \in L^2(R^2)$. Hence,

$$f(P\cdot) = \sum_{j\in Z, k\in Z^2} f_{j,k}2^{-\frac{j}{2}}\psi(M_1^{-j}\cdot - k) = \sum_{j\in Z, k\in Z^2} f_{j,k}2^{-\frac{j}{2}}\tilde{\psi}(PM_1^{-j}\cdot - Pk)$$

for some $\{f_{j,k}\} \in l^2(Z\times Z^2)$, and consequently,

$$f(\cdot) = \sum_{j\in Z, k\in Z^2} f_{j,k}2^{-\frac{j}{2}}\tilde{\psi}(PM_1^{-j}P^{-1}\cdot - Pk) = \sum_{j\in Z, k\in Z^2} f_{j,k}2^{-\frac{j}{2}}\tilde{\psi}(M_2^{-j}\cdot - P^{-1}k).$$

Therefore, $\tilde{\psi}$ is a M_2-wavelet for $L^2(R^2)$. The necessity is proved. □

The sufficiency can be proved analogously.

Remark 2.1. By the same procedure as that in Lemma 2.1, we have: Assume that $\psi \in L^2(R^2)$, that M_1 is a 2×2 dilation matrix with $M_1^2 = 2I$, and that M_2 is similar to M_1, i.e., $M_2 = PM_1P^{-1}$ for some invertible matrix P. Define $\tilde{\psi}(\cdot) = |\det P|^{-\frac{1}{2}}\psi(P^{-1}\cdot)$. Then $\psi(\cdot)$ is an M_1-wavelet for $L^2(R^2)$ if and only if $\{2^{-\frac{j}{2}}\tilde{\psi}(M_2^{-j}\cdot - Pk)\colon j\in Z, k\in Z^2\}$ is an orthonormal basis for $L^2(R^2)$.

Lemma 2.2. *A 2×2 integer matrix M is a dilation matrix with $M^2 = 2I$ if and only if $M = \pm\begin{pmatrix} k & b \\ c & -k\end{pmatrix}$ for some $k\in Z_+$, and some b, $c\in Z$ satisfying $bc = 2-k^2$.*

Proof. Suppose $M = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$. Obviously the eigenvalues of M are larger than 1 in modulus if $M^2 = 2I$. So it suffices to show that $M^2 = 2I$ if and only if $M = \pm\begin{pmatrix} k & b \\ c & -k\end{pmatrix}$ for some $0\leqslant k\in Z$, and some b, $c\in Z$ satisfying $bc = 2-k^2$. It is easy to check that the fact that $M^2 = 2I$ is equivalent to the fact that

$$a^2 + bc = d^2 + bc = 2, \qquad (2.2)$$

$$ab + bd = ac + cd = 0. \qquad (2.3)$$

Since $a, d \in Z$, it follows from (2.2) that $bc \neq 0$. Hence, (2.2) and (2.3) are equivalent to the fact that

$$a^2 + bc = 2 \quad \text{and} \quad a = -d,$$

which completes the proof. $\square$

Lemma 2.3. (1) *Integral similarity is an equivalent relation*;

(2) *Two* 2×2 *matrices* A *and* B *are integrally similar if and only if* $-A$ *and* $-B$ *are integrally similar*;

(3) *Let* A *and* B *be two* 2×2 *integrally similar matrices. If* A *is a dilation matrix with* $A^2 = 2I$, *then so does* B;

(4) *If a* 2×2 *matrix* A *is integrally similar to the matrix* $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, *then so does* $-A$.

Proof. We only give the proof of (4). The proofs of the others are omitted. Suppose $PAP^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ for some integer matrix with $|\det P| = 1$. Putting $Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} P$, we obtain that $Q(-A)Q^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. The proof is completed. $\square$

Remark 2.2. Proposition 2.1 in [4] and Lemma 5.2 in [15] give a classification of integer matrices with determinant ± 2 in terms of integral similarity. By Lemma 2.3, they are equivalent, and any dilation matrix M with $M^2 = 2I$ is integrally similar to $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. It follows from Lemmas 2.2 and 2.3(4) that, for dilation matrices M with $M^2 = 2I$, the question of looking for an integrally similar matrix P from M to $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is reduced to looking for an integrally similar matrix P from $\begin{pmatrix} k & b \\ c & -k \end{pmatrix}$ to $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, where $k \in Z_+$, $b, c \in Z$, $bc = 2 - k^2$.

Lemma 2.4. *Let* $k \in Z_+$, $b, c \in Z$, *and* $bc = 2 - k^2$. *Then* P *is an integrally similar matrix from* $\begin{pmatrix} k & b \\ c & -k \end{pmatrix}$ *to* $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ *if and only if*

$$P = \begin{pmatrix} (k+1)x + cy & bx - (k-1)y \\ x & y \end{pmatrix},$$

where $x, y \in Z$, *and*

$$(bx^2 - 2kxy - cy^2)^2 = 1. \tag{2.4}$$

Proof. An unknown matrix $\begin{pmatrix} \alpha & \beta \\ x & y \end{pmatrix}$ satisfies

$$\begin{pmatrix} \alpha & \beta \\ x & y \end{pmatrix} \begin{pmatrix} k & b \\ c & -k \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ x & y \end{pmatrix}$$

if and only if

$$\begin{cases} \alpha - (k+1)x - cy = 0, \\ \beta - bx + (k-1)y = 0, \\ (k-1)\alpha + c\beta - x = 0, \\ b\alpha - (k+1)\beta - y = 0. \end{cases} \tag{2.5}$$

It follows from simple computation that (2.5) is equivalent to

$$\begin{cases} \alpha = (k+1)x + cy, \\ \beta = bx - (k-1)y. \end{cases} \tag{2.6}$$

Hence, $\begin{pmatrix} k & b \\ c & -k \end{pmatrix}$ is integrally similar to $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ if and only if there exist $x, y \in Z$ such that $\left[\det\begin{pmatrix} \alpha & \beta \\ x & y \end{pmatrix}\right]^2 = 1$, where α, β satisfy (2.6). It is easy to check that $\left[\det\begin{pmatrix} \alpha & \beta \\ x & y \end{pmatrix}\right]^2 = (bx^2 - 2kxy - cy^2)^2$. Therefore, $\begin{pmatrix} k & b \\ c & -k \end{pmatrix}$ is integrally similar to $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ if and only if the bivariate equation

$$(bx^2 - 2kxy - cy^2)^2 = 1$$

has an integer solution. The proof is completed. $\square$

It follows from Lemma 2.4 that looking for an integrally similar matrix P, from a 2×2 dilation M with $M^2 = 2I$ to $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, is reduced to looking for an integer solution to the bivariate equation (2.4). Obviously it is not easy. Here we give an algorithm to get such a P. The idea is borrowed from [15].

(I) By Lemma 2.2, $M = \pm\begin{pmatrix} k & b \\ c & -k \end{pmatrix}$ for some $k \in Z_+$, and some $b, c \in Z$ satisfying $bc = 2 - k^2$. Note that $Q\begin{pmatrix} -k & -b \\ -c & k \end{pmatrix}Q^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ if $P\begin{pmatrix} k & b \\ c & -k \end{pmatrix}P^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, where $Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}P$. So it suffices to look for the integrally similar matrix P from $\begin{pmatrix} k & b \\ c & -k \end{pmatrix}$ to $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, where $k \in Z_+$, and $b, c \in Z$ with $bc = 2 - k^2$.

(II) If $k = 0$, then $M = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$, or $\begin{pmatrix} 0 & -2 \\ -1 & 0 \end{pmatrix}$, and we take $P = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, or $\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$, respectively.

If $k = 1$, then $M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ or $\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$, and we take $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, respectively.

If $k = 2$, then $M = \begin{pmatrix} 2 & 1 \\ -2 & -2 \end{pmatrix}$, $\begin{pmatrix} 2 & -1 \\ 2 & -2 \end{pmatrix}$, $\begin{pmatrix} 2 & 2 \\ -1 & -2 \end{pmatrix}$, or $\begin{pmatrix} 2 & -2 \\ 1 & -2 \end{pmatrix}$, and we take $P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$, or $\begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix}$, respectively.

(III) $k > 2$.

Since $k \in Z$, $bc = 2 - k^2 \neq 0$. If $|b| \geqslant k$ and $|c| \geqslant k$, then $|b| = \frac{k^2-2}{|c|} \leqslant \frac{k^2-2}{k} = k - \frac{2}{k}$. Noting $b \in Z$ and $0 < \frac{2}{k} < 1$, we obtain that $0 < |b| < k$ or $0 < |c| < k$.

When $0 < |b| < k$, it is easy to check that

$$\begin{pmatrix} 1 & 0 \\ \operatorname{sign}(b) & 1 \end{pmatrix}\begin{pmatrix} k & b \\ c & -k \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \operatorname{sign}(b) & 1 \end{pmatrix}^{-1} = \begin{pmatrix} k - |b| & b \\ 2k\operatorname{sign}(b) + c - b & |b| - k \end{pmatrix}.$$

When $0 < |c| < k$, it is easy to check that

$$\begin{pmatrix} 1 & -\operatorname{sign}(c) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} k & b \\ c & -k \end{pmatrix}\begin{pmatrix} 1 & -\operatorname{sign}(c) \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} k - |c| & 2k\operatorname{sign}(c) - c + b \\ c & |c| - k \end{pmatrix}.$$

In view of Lemma 2.3 (3), applying the above procedure to $\begin{pmatrix} k & b \\ c & -k \end{pmatrix}$ finitely many times, we reduces (III) to (II). Then the product of all these similar matrices gives the P.

3. Some properties of a special M and M-wavelets

Throughout this section M is always referred to be the matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

Lemma 3.1. *Let* $M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. *For* $c>0$, *define*

$$S_c^{(1)} = \{(\xi_1, \xi_2)^T\colon\ |\xi_1| \leqslant c, |\xi_1 + \xi_2| \leqslant c\},$$
$$S_c^{(2)} = \{(\xi_1, \xi_2)^T\colon\ |\xi_2| \leqslant c, |\xi_1 - \xi_2| \leqslant c\}.$$

Then

$$M^{j_1} S_c^{(1)} \subset M^{j_2} S_c^{(1)} \quad \text{and} \quad M^{j_1} S_c^{(2)} \subset M^{j_2} S_c^{(2)}$$

for $j_1 < j_2,\ j_1, j_2 \in Z$.

The proof is omitted.

Lemma 3.2. *Let* $M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $c>0$, $S_c^{(1)}$ *and* $S_c^{(2)}$ *be defined as in Lemma* 3.1. *Define*

$$I_c^{(1)} = \{(\xi_1, \xi_2)^T\colon\ |\xi_1| \leqslant c, c \leqslant |\xi_1 + \xi_2| \leqslant 2c\},$$
$$I_c^{(2)} = \{(\xi_1, \xi_2)^T\colon\ |\xi_2| \leqslant c, c \leqslant |\xi_1 - \xi_2| \leqslant 2c\}.$$

Then,

(1) *Both* $\{M^j I_c^{(1)}\colon\ j \in Z\}$ *and* $\{M^j I_c^{(2)}\colon\ j \in Z\}$ *are partitions of* R^2;
(2) $\{M^j I_c^{(1)}\colon\ j \in Z, j<0\}$ *and* $\{M^j I_c^{(2)}\colon\ j \in Z, j<0\}$ *are partitions of* $S_c^{(1)}$ *and* $S_c^{(2)}$, *respectively*.

Proof. We only give the proof of the case $I_c^{(1)}$. The proofs of the other parts can be given analogously. Define $S = I_c^{(1)} \cup M I_c^{(1)}$, then

$$S = \{(\xi_1, \xi_2)^T\colon\ |\xi_1| \leqslant c, c \leqslant |\xi_1 + \xi_2| \leqslant 2c\}$$
$$\cup \{(\xi_1, \xi_2)^T\colon\ c \leqslant |\xi_1| \leqslant 2c, |\xi_1 + \xi_2| \leqslant 2c\}.$$

It is easy to check that $\{2^j S\colon\ j \in Z\}$ is a partition of R^2. It is obvious that $I_c^{(1)} \cap M^{2j} I_c^{(1)} \subset S \cap 2^j S$ for $0 \neq j \in Z$, and that $I_c \cap M^{2j+1} I_c \subset S \cap 2^j M I_c \subset S \cap 2^j S$ for $j \in Z$.

Hence,

$$|I_c^{(1)} \cap M^j I_c^{(1)}| = 0 \tag{3.1}$$

for $0 \neq j \in Z$.

Since $\{2^j S\colon j\in Z\}$ is a partition of R^2,

$$\begin{aligned} R^2 &= \bigcup_{j\in Z} 2^j S \\ &= \left(\bigcup_{j\in Z} 2^j I_c^{(1)}\right) \cup \left(\bigcup_{j\in Z} 2^j M I_c^{(1)}\right) \\ &= \left(\bigcup_{j\in Z} M^{2j} I_c^{(1)}\right) \cup \left(\bigcup_{j\in Z} M^{2j+1} I_c^{(1)}\right) \\ &= \bigcup_{j\in Z} M^j I_c^{(1)} \end{aligned}$$

up to a set of measure 0. This together with (3.1) implies that $\{M^j I_c^{(1)}\colon j\in Z\}$ is a partition of R^2.

It is obvious that $I_c^{(1)} \subset M S_c^{(1)}$, and consequently, $M^j I_c^{(1)} \subset M^{j+1} S_c^{(1)}$ for $j\in Z$. So, by Lemma 3.1, $M^j I_c^{(1)} \subset M^{j+1} S_c^{(1)} \subset S_c^{(1)}$ for $j<0$, $j\in Z$.

Hence,

$$\bigcup_{j<0} M^j I_c^{(1)} \subset S_c^{(1)}. \tag{3.2}$$

For any $\xi\in S_c^{(1)}$ with $\xi_1 \neq 0$, there exists $j_1 \leqslant 0$ such that $2^{j_1-1}c \leqslant |\xi_1| \leqslant 2^{j_1}c$. Since

$$M I_c^{(1)} = \{(\xi_1,\xi_2)^T\colon\ c\leqslant|\xi_1|\leqslant 2c, |\xi_1+\xi_2|\leqslant 2c\},$$

$\xi\in 2^{j_1-1} M I_c^{(1)} = M^{2j_1-1} I_c^{(1)}$ when $|\xi_1+\xi_2|\leqslant 2^{j_1}c$. When $|\xi_1+\xi_2| > 2^{j_1}c$, $2^{j_2-1}c \leqslant |\xi_1 + \xi_2| \leqslant 2^{j_2}c$ for some $j_1 \leqslant j_2 - 1$. Noting that $|\xi_1| \leqslant 2^{j_2-1}c$, we obtain that $\xi\in 2^{j_2-1} I_c^{(1)} = M^{2j_2-2} I_c^{(1)}$. Therefore,

$$S_c^{(1)} \subset \bigcup_{j<0} M^j I_c^{(1)} \tag{3.3}$$

up to a set of Lebesgue measure zero. This together with (3.1) and (3.2) implies that $\{M^j I_c^{(1)}\colon j\in Z, j<0\}$ is a partition of $S_c^{(1)}$. The proof is completed. $\square$

Lemma 3.3. *Let* $M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $0<c\leqslant\pi$, $I_c^{(1)}$ *and* $I_c^{(2)}$ *be defined as in Lemma* 3.2, Q_1, Q_2, Q_3, *and* Q_4 *denote the four quadrants in* R^2. *Define* $\psi_i^{(1)}$ *and* $\psi_i^{(2)}$ *by*

$$\widehat{\psi_i^{(1)}} = \frac{1}{2\pi}\chi_{Q_i\cap I_c^{(1)}} \quad \text{and} \quad \widehat{\psi_i^{(2)}} = \frac{1}{2\pi}\chi_{Q_i\cap I_c^{(2)}}$$

for $1\leqslant i\leqslant 4$. *Then both the collection*

$$\{\psi_1^{(1)}, \psi_2^{(1)}, \psi_3^{(1)}, \psi_4^{(1)}\}$$

and the collection

$$\{\psi_1^{(2)}, \psi_2^{(2)}, \psi_3^{(2)}, \psi_4^{(2)}\}$$

generate M*-tight frames for* $L^2(R^2)$ *with frame bound* 1.

Proof. We only give the proof of $\{\psi_1^{(1)},\ \psi_2^{(1)},\ \psi_3^{(1)},\ \psi_4^{(1)}\}$. The proof of the other is analogous. Taking the closed subspace of $L^2(R^2)$

$$\mathscr{D} = \{f \in L^2(R^2)\colon \hat{f} \in L^\infty(R^2), \operatorname{supp}(\hat{f}) \subset K \text{ for some compact } K \subset R^2 \backslash \{0\}\},$$

then it suffices to show that

$$||f||^2 = \sum_{i=1}^{4} \sum_{j \in Z, k \in Z^2} |\langle f, \psi_{i,j,k}^{(1)} \rangle|^2 \tag{3.4}$$

for $f \in \mathscr{D}$.

By [3, Corollary 3.3], it suffices to show that

$$\sum_{i=1}^{4} \sum_{j \in Z} |\widehat{\psi_i^{(1)}}(M^j \xi)|^2 = \frac{1}{4\pi^2} \tag{3.5}$$

for a.e. $\xi \in R^2$, and

$$\sum_{i=1}^{4} \sum_{j=0}^{\infty} \widehat{\psi_i^{(1)}}(M^j \xi) \overline{\widehat{\psi_i^{(1)}}(M^j(\xi + 2\pi k))} = 0 \tag{3.6}$$

for $k \in Z^2 \backslash MZ^2$ and a.e. $\xi \in R^2$.

By Lemma 3.2, $\{M^j I_c^{(1)}\colon j \in Z\}$ is a partition of R^2. Hence, by the definition of $\psi_i^{(1)}$, (3.5) holds. Since the diameter of the sets $Q_i \cap I_c^{(1)}$ measured along the coordinate axes is not larger than 2π, (3.6) holds. The proof is completed. □

Lemma 3.4. *Define*

$$S_\pi^{(1)} = \{(\xi_1, \xi_2)^T\colon |\xi_1| \leqslant \pi, |\xi_1 + \xi_2| \leqslant \pi\},$$
$$S_\pi^{(2)} = \{(\xi_1, \xi_2)^T\colon |\xi_2| \leqslant \pi, |\xi_1 - \xi_2| \leqslant \pi\}.$$

Then,

$$\sum_{k \in Z^2} \left| \int_{S_\pi^{(1)}} d\xi\, f(\xi) e^{-i\langle k, \xi \rangle} \right|^2 = 4\pi^2 \int_{S_\pi^{(1)}} d\xi\, |f(\xi)|^2 \quad \textit{for} \quad f \in L^2(S_\pi^{(1)}), \tag{3.7}$$

$$\sum_{k \in Z^2} \left| \int_{S_\pi^{(2)}} d\xi\, f(\xi) e^{-i\langle k, \xi \rangle} \right|^2 = 4\pi^2 \int_{S_\pi^{(2)}} d\xi\, |f(\xi)|^2 \quad \textit{for} \quad f \in L^2(S_\pi^{(2)}). \tag{3.8}$$

Proof. We only prove (3.7). Eq. (3.8) can be proved analogously. Since the diameter of $S_\pi^{(1)}$ measured along the coordinate axes is exactly 2π, f can be $2\pi Z^2$-periodically extended. We still denote the extended function by f.

Define

$$R_1 = \{(\xi_1,\xi_2)^T:\ -\pi \leqslant \xi_1 \leqslant 0, -\pi-\xi_1 \leqslant \xi_2 \leqslant \pi\}$$
$$\cup \{(\xi_1,\xi_2)^T:\ 0 \leqslant \xi_1 \leqslant \pi, -\pi \leqslant \xi_2 \leqslant \pi-\xi_1\},$$

$$R_2 = \{(\xi_1,\xi_2)^T:\ -\pi \leqslant \xi_1 \leqslant 0, \pi \leqslant \xi_2 \leqslant \pi-\xi_1\},$$
$$R_3 = \{(\xi_1,\xi_2)^T:\ 0 \leqslant \xi_1 \leqslant \pi, -\pi-\xi_1 \leqslant \xi_2 \leqslant -\pi\}.$$

Then,

$$\int_{S_\pi^{(1)}} d\xi f(\xi) e^{-i\langle k,\xi\rangle} = \sum_{i=1}^{3} \int_{R_i} d\xi f(\xi) e^{-i\langle k,\xi\rangle}. \tag{3.9}$$

$$\begin{aligned}\int_{R_2} & d\xi f(\xi) e^{-i\langle k,\xi\rangle} \\ &= \int_{R_2-(0,2\pi)^T} d\eta f(\eta+(0,2\pi)^T) e^{-i\langle k,\eta\rangle} \\ &= \int_{\{(\eta_1,\eta_2)^T:\ -\pi\leqslant\eta_1\leqslant 0, -\pi\leqslant\eta_2\leqslant-\pi-\eta_1\}} d\eta f(\eta) e^{-i\langle k,\eta\rangle} \\ &= \int_{\{(\xi_1,\xi_2)^T:\ -\pi\leqslant\xi_1\leqslant 0, -\pi\leqslant\xi_2\leqslant-\pi-\xi_1\}} d\xi f(\xi) e^{-i\langle k,\xi\rangle}. \end{aligned} \tag{3.10}$$

Analogously,

$$\int_{R_3} d\xi f(\xi) e^{-i\langle k,\xi\rangle} = \int_{\{(\xi_1,\xi_2)^T:\ 0\leqslant\xi_1\leqslant\pi, \pi-\xi_1\leqslant\xi_2\leqslant\pi\}} d\xi f(\xi) e^{-i\langle k,\xi\rangle}. \tag{3.11}$$

This together with (3.9) and (3.10) yields that

$$\int_{S_\pi^{(1)}} d\xi f(\xi) e^{-i\langle k,\xi\rangle} = \int_{[-\pi,\pi]^2} d\xi f(\xi) e^{-i\langle k,\xi\rangle},$$

and consequently,

$$\begin{aligned}\sum_{k\in Z^2} \left| \int_{S_\pi^{(1)}} d\xi f(\xi) e^{-i\langle k,\xi\rangle} \right|^2 &= \sum_{k\in Z^2} \left| \int_{[-\pi,\pi]^2} d\xi f(\xi) e^{-i\langle k,\xi\rangle} \right|^2 \\ &= 4\pi^2 \int_{[-\pi,\pi]^2} d\xi\, |f(\xi)|^2. \end{aligned}$$

By the arguments similar to that of the above, we obtain that

$$\int_{S_\pi^{(1)}} d\xi\, |f(\xi)|^2 = \int_{[-\pi,\pi]^2} d\xi\, |f(\xi)|^2.$$

Therefore,

$$\sum_{k\in Z^2}\left|\int_{S_\pi^{(1)}} d\xi\, f(\xi)e^{-i\langle k,\xi\rangle}\right|^2 = 4\pi^2\int_{S_\pi^{(1)}} d\xi\,|f(\xi)|^2.$$

The proof is completed. □

Lemma 3.5. *Let* $M=\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$, $0<c\leqslant\pi$, $S_c^{(1)}$, $I_c^{(1)}$, $S_c^{(1)}$, *and* $I_c^{(2)}$ *be defined as in Lemmas* 3.1 *and* 3.2, *respectively. Suppose* ψ *is an* M*-wavelet for* $L^2(R^2)$. *Then,*

$$\sum_{j<0}2^{-j}\int_{M^jI_c^{(1)}} d\xi\,|\hat\psi(\xi)|^2\leqslant\frac{c^2}{\pi^2}\leqslant\sum_{j<0}2^{-j}\int_{M^jI_c^{(1)}} d\xi\,|\hat\psi(\xi)|^2 + \int_{R^2\setminus S_c^{(1)}} d\xi\,|\hat\psi(\xi)|^2, \tag{3.12}$$

$$\sum_{j<0}2^{-j}\int_{M^jI_c^{(2)}} d\xi\,|\hat\psi(\xi)|^2\leqslant\frac{c^2}{\pi^2}\leqslant\sum_{j<0}2^{-j}\int_{M^jI_c^{(2)}} d\xi\,|\hat\psi(\xi)|^2 + \int_{R^2\setminus S_c^{(2)}} d\xi\,|\hat\psi(\xi)|^2. \tag{3.13}$$

In addition,

$$\sum_{j<0}(2^{-j}-1)\int_{M^jI_c^{(1)}} d\xi\,|\hat\psi(\xi)|^2=\frac{c^2}{\pi^2}-1\quad \textit{if}\ \ \mathrm{supp}(\hat\psi)\subset MS_c^{(1)}, \tag{3.14}$$

$$\sum_{j<0}(2^{-j}-1)\int_{M^jI_c^{(2)}} d\xi\,|\hat\psi(\xi)|^2=\frac{c^2}{\pi^2}-1\quad \textit{if}\ \ \mathrm{supp}(\hat\psi)\subset MS_c^{(2)}. \tag{3.15}$$

Proof. We only prove (3.12) and (3.14). Eqs. (3.13) and (3.15) can be proved analogously. Suppose $\psi_i^{(1)}$, $1\leqslant i\leqslant 4$, are defined as in Lemma 3.3, then $||\psi_i^{(1)}||_2^2 = ||\widehat{\psi_i^{(1)}}||_2^2=\frac{c^2}{4\pi^2}$. Since ψ is an M-wavelet for $L^2(R^2)$,

$$\begin{aligned}\frac{c^2}{\pi^2} &= \sum_{i=1}^4 ||\psi_i^{(1)}||_2^2\\ &= \sum_{i=1}^4\sum_{j\in Z,k\in Z^2}|\langle\psi_i^{(1)},\psi_{j,k}\rangle|^2\\ &= \sum_{i=1}^4\sum_{j\in Z,k\in Z^2}2^j\left|\int_{R^2} d\xi\,\overline{\widehat{\psi_i^{(1)}}(\xi)}\hat\psi(M^j\xi)e^{-i\langle M^jk,\xi\rangle}\right|^2\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{4} \sum_{j\in Z, k\in Z^2} 2^{-j}\left|\int_{R^2} d\xi\, \widehat{\psi_i^{(1)}}(M^{-j}\xi)\hat{\psi}(\xi)e^{-i\langle k,\xi\rangle}\right|^2 \\
&= \sum_{i=1}^{4} \sum_{j<0, k\in Z^2} 2^{-j}\left|\int_{R^2} d\xi\, \widehat{\psi_i^{(1)}}(M^{-j}\xi)\hat{\psi}(\xi)e^{-i\langle k,\xi\rangle}\right|^2 \\
&\quad + \sum_{i=1}^{4} \sum_{j\geqslant 0, k\in Z^2} 2^{-j}\left|\int_{R^2} d\xi\, \widehat{\psi_i^{(1)}}(M^{-j}\xi)\hat{\psi}(\xi)e^{-i\langle k,\xi\rangle}\right|^2 \\
&\equiv J' + J''. \qquad (3.16)
\end{aligned}
$$

It is easy to check that $\mathrm{supp}(\widehat{\psi_i^{(1)}}(M^{-j}\cdot)) \subset M^{j+1}S_c^{(1)}$ for $j\in Z$, so, by Lemma 3.1, $\mathrm{supp}(\widehat{\psi_i^{(1)}}(M^{-j}\cdot)) \subset S_c^{(1)} \subset S_\pi^{(1)}$ for $j<0$, $j\in Z$. Applying Lemma 3.4, we obtain that

$$
\begin{aligned}
J' &= \sum_{i=1}^{4} \sum_{j<0, k\in Z^2} 2^{-j}\left|\int_{S_\pi} d\xi\, \widehat{\psi_i^{(1)}}(M^{-j}\xi)\hat{\psi}(\xi)e^{-i\langle k,\xi\rangle}\right|^2 \\
&= \sum_{i=1}^{4} \sum_{j<0} 2^{-j}4\pi^2 \int_{S_\pi^{(1)}} d\xi\, |\widehat{\psi_i^{(1)}}(M^{-j}\xi)\hat{\psi}(\xi)|^2 \\
&= \sum_{i=1}^{4} \sum_{j<0} 2^{-j} \int_{M^j(Q_i\cap I_c^{(1)})} d\xi\, |\hat{\psi}(\xi)|^2 \\
&= \sum_{j<0} 2^{-j} \int_{M^j I_c^{(1)}} d\xi\, |\hat{\psi}(\xi)|^2. \qquad (3.17)
\end{aligned}
$$

$$
J'' \leqslant \sum_{i=1}^{4} \sum_{j\geqslant 0, k\in Z^2} 2^{-j}\left|\int_{R^2} d\xi\, \widehat{\psi_i^{(1)}}(M^{-j}\xi)\hat{\psi}(\xi)e^{-i\langle M^{-j}k,\xi\rangle}\right|^2
$$

since the right side contains more terms than the left side. By Lemma 3.1, it is easy to check that the equality holds when $\mathrm{supp}(\hat{\psi}) \subset MS_c^{(1)}$.

Taking g such that $\hat{g} = (1-\chi_{S_c^{(1)}})\bar{\hat{\psi}}$, then, by Lemma 3.3, we obtain that

$$
\begin{aligned}
J'' &\leqslant \sum_{i=1}^{4} \sum_{j\geqslant 0, k\in Z^2} 2^{-j}\left|\int_{R^2} d\xi\, \widehat{\psi_i^{(1)}}(M^{-j}\xi)\hat{\psi}(\xi)e^{-i\langle M^{-j}k,\xi\rangle}\right|^2 \\
&= \sum_{i=1}^{4} \sum_{j\in Z, k\in Z^2} 2^{-j}\left|\int_{R^2} d\xi (1-\chi_{S_c^{(1)}})\hat{\psi}(\xi)\widehat{\psi_i^{(1)}}(M^{-j}\xi)e^{-i\langle M^{-j}k,\xi\rangle}\right|^2 \\
&= \sum_{i=1}^{4} \sum_{j\in Z, k\in Z^2} |\langle g, \psi_{i,j,k}^{(1)}\rangle|^2
\end{aligned}
$$

$$= \int_{R^2} d\xi \, |\hat{g}(\xi)|^2$$

$$= \int_{R^2 \setminus S_c^{(1)}} d\xi \, |\hat{\psi}(\xi)|^2, \tag{3.18}$$

where Lemma 3.3 is used in the third equality, the inequality sign, by Lemma 3.1, changes to be equality sign when $\text{supp}(\hat{\psi}) \subset MS_c^{(1)}$.

It follows from (3.16)–(3.18) that

$$\sum_{j<0} 2^{-j} \int_{M^j I_c} d\xi \, |\hat{\psi}(\xi)|^2 \leqslant \frac{c^2}{\pi^2} \leqslant \sum_{j<0} 2^{-j} \int_{M^j I_c} d\xi \, |\hat{\psi}(\xi)|^2 + \int_{R^2 \setminus S_c^{(1)}} d\xi \, |\hat{\psi}(\xi)|^2,$$

and when $\text{supp}(\hat{\psi}) \subset MS_c^{(1)}$,

$$\frac{c^2}{\pi^2} = \sum_{j<0} 2^{-j} \int_{M^j I_c} d\xi \, |\hat{\psi}(\xi)|^2 + \int_{R^2 \setminus S_c^{(1)}} d\xi \, |\hat{\psi}(\xi)|^2$$

$$= \sum_{j<0} 2^{-j} \int_{M^j I_c} d\xi \, |\hat{\psi}(\xi)|^2 + 1 - \int_{S_c^{(1)}} d\xi \, |\hat{\psi}(\xi)|^2. \tag{3.19}$$

By Lemma 3.2, (3.19) is equivalent to

$$\sum_{j<0} (2^{-j} - 1) \int_{M^j I_c} d\xi \, |\hat{\psi}(\xi)|^2 = \frac{c^2}{\pi^2} - 1.$$

The proof is completed. □

4. Proofs of Theorems

Assume that Theorems 1.1 and 1.2 hold for $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$-wavelets. For any M-wavelet ψ, $\tilde{\psi}(\cdot) = \psi(P^{-1}\cdot)$ is an $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$-wavelet by Lemma 2.1. Applying the assumption to $\tilde{\psi}$, we obtain that Theorems 1.1 and 1.2. Therefore, it suffices to show that Theorems 1.1 and 1.2 hold for $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$-wavelets.

Proof of Theorem 1.1. By contradiction. Let $M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, and ψ be an M-wavelet. Then we can take $P = I$. For $c > 0$, define $S_c^{(1)}$, $S_c^{(2)}$, $I_c^{(1)}$, and $I_c^{(2)}$ as in Lemmas 3.1 and 3.2. Suppose that $|\text{supp}(\hat{\psi}) \cap S_c^{(1)}| = 0$ (or $|\text{supp}(\hat{\psi}) \cap S_c^{(2)}| = 0$). Then, by (2) of Lemma 3.2, $|\text{supp}(\hat{\psi}) \cap M^j I_c^{(1)}| = 0$ (or $|\text{supp}(\hat{\psi}) \cap M^j I_c^{(2)}| = 0$) for $j < 0$. Applying (3.12) (or (3.13)) of Lemma 2.6 to ψ, we obtain that

$$\frac{c^2}{\pi^2} \leqslant \int_{R^2 \setminus S_c^{(1)}} d\xi \, |\hat{\psi}(\xi)|^2 = \int_{R^2} d\xi \, |\hat{\psi}(\xi)|^2 = 1$$

$$\left(\text{or} \quad \frac{c^2}{\pi^2} \leqslant \int_{R^2 \setminus S_c^{(2)}} d\xi \, |\hat{\psi}(\xi)|^2 = \int_{R^2} d\xi \, |\hat{\psi}(\xi)|^2 = 1 \right),$$

which implies $c \leqslant \pi$. The proof is completed. □

Proof Theorem 1.2. Let $M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, and ψ be an M-wavelet. Then we can take $P = I$. Define $S_\pi^{(1)}$, $S_\pi^{(2)}$, $I_\pi^{(1)}$, and $I_\pi^{(2)}$ as in Lemmas 3.1 and 3.2. Since $K \subset MS_\pi^{(1)}$ (or $K \subset MS_\pi^{(2)}$), (3.14) (or (3.15)) of Lemma 3.5 takes the form

$$0 = \frac{1}{4\pi^2} \sum_{j<0} (2^{-j} - 1)|K \cap M^j I_\pi^{(1)}| \text{ (or } 0 = \frac{1}{4\pi^2} \sum_{j<0} (2^{-j} - 1)|K \cap M^j I_\pi^{(2)}|). \tag{4.1}$$

Again by Lemma 3.2, we obtain that

$$|K \cap S_\pi^{(1)}| = \left| K \cap \left(\bigcup_{j<0} M^j I_\pi^{(1)} \right) \right| \leqslant \sum_{j<0} (2^{-j} - 1)|K \cap M^j I_\pi^{(1)}| = 0$$

$$\left(\text{or } K \cap S_\pi^{(2)}| = \left| K \cap \left(\bigcup_{j<0} M^j I_\pi^{(2)} \right) \right| \leqslant \sum_{j<0} (2^{-j} - 1)|K \cap M^j I_\pi^{(2)}| = 0 \right).$$

Hence, $|K \cap S_\pi^{(1)}| = 0$ (or $|K \cap S_\pi^{(1)}| = 0$). The proof is completed. □

In the next, we give an example of Theorem 1.2.

Example 2.1. Let a matrix M be integrally similar to the matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, i.e., $PMP^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ for some integer matrix P with $|\det P| = 1$. Define $I_\pi^{(1)}$ and $I_\pi^{(2)}$ as in Lemma 3.2, define ψ by

$$\hat{\psi}(\cdot) = -\frac{1}{2\pi} e^{-i\frac{1}{2}\langle P^{-1}\binom{1}{1}, \cdot\rangle} \chi_{P^* I_\pi^{(1)}}(\cdot)$$

$$\text{(or } \hat{\psi}(\cdot) = -\frac{1}{2\pi} e^{-i\frac{1}{2}\langle P^{-1}\binom{1}{1}, \cdot\rangle} \chi_{P^* I_\pi^{(2)}}(\cdot)).$$

Then ψ is an M-wavelet for $L^2(R^2)$.

Proof. By Lemma 2.1, it suffices to show the proposition is true for $M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Suppose $M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, we can take $P = I$. We denote by $m_0(\cdot)$ the $2\pi Z$ periodization of $\chi_{[-\frac{\pi}{2}, \frac{\pi}{2}]}$. Then m_0 generates an orthonormal MRA for $L^2(R)$ with m_0 being its symbol. Define $H_0(\xi) = m_0(\xi_1)$ (or $H_0(\xi) = m_0(\xi_2)$) for $\xi \in R^2$. By [6], H_0 generates an orthonormal MRA for $L^2(R^2)$ with H_0 being its symbol and ϕ being its scaling function, where

$$\hat{\phi}(\cdot) = \frac{1}{2\pi} \prod_{j=1}^{\infty} H_0(M^{-j}\cdot).$$

Then the basic wavelet $\tilde{\psi}$ can be defined by

$$\hat{\tilde{\psi}}(\cdot) = H_1(M^{-1}\cdot)\hat{\phi}(M^{-1}\cdot),$$

where $H_1(\xi) = -e^{-i\xi_1}\overline{H_0(\xi + (\pi, \pi)^T)}$. It is easy to check that

$$\hat{\tilde{\psi}}(\xi) = -\frac{1}{2\pi} e^{-i\frac{\xi_1+\xi_2}{2}} \chi_{I_\pi^{(1)}}(\xi) \ (\text{or } \hat{\tilde{\psi}}(\xi) = -\frac{1}{2\pi} e^{-i\frac{\xi_1+\xi_2}{2}} \chi_{I_\pi^{(2)}}(\xi))$$

for $\xi \in R^2$. Hence, $\psi = \tilde{\psi}$, and thus ψ is an M-wavelet for $L^2(R^2)$. The proof is completed. □

Acknowledgments

The author would like to thank the referees for their helpful suggestions, which greatly improved the quality of this paper.

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